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On the maximum number of Gel'fand patterns of given weight

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Abstract. If $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_N]$ is a partition of n, it is proved in this note that the maximum number of Gel'fand patterns of $[\lambda]$ corresponding to any weight is equal to $f_{[\lambda]}$, where $f_{[\lambda]}$ is the number of standard Young tableaux corresponding to the partition.

1. Gel'fand patterns

Gel'fand patterns form an ingenious device for constructing a basis (Nagel and Moshinsky 1965) for the irreducible representations of U(N) in the chain

$$\mathrm{U}(N) \supset \mathrm{U}(N-1) \supset \ldots \supset \mathrm{U}(1).$$

They are widely used in the group theoretical applications to problems in nuclear and particle physics (Kramer and Moshinsky 1968, O'Raifeartaigh 1968). Also, they form a basic tool in the recent studies on pattern calculus (Holman and Biedenharn 1971) and a knowledge of the maximum number of Gel'fand patterns for a given weight is of considerable interest.

Let $\lambda_1, \lambda_2, \ldots \lambda_N$ be a set of non-negative integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N$. Suppose $\lambda_1 + \lambda_2 + \ldots + \lambda_N = n$. $\lambda_1, \lambda_2, \ldots, \lambda_N$ may be regarded as a partition of n including zero parts in the partition. For the sake of convenience we write $\lambda_1, \lambda_2, \ldots, \lambda_N$ as $m_{1N}, m_{2N}, \ldots, m_{NN}$ respectively. The following arrangement

$$\begin{pmatrix} [m] \\ \\ \\ (m) \end{pmatrix} = \left| \begin{array}{ccccccc} m_{1N} & & m_{2N} & \cdots\cdots\cdots\cdots & & & m_{NN} \\ & m_{1N-1} & & m_{2N-1} \cdots & m_{N-1N-1} & & \\ & & \cdots\cdots\cdots\cdots\cdots & & & & \\ & & & \cdots\cdots\cdots & & & \\ & & & m_{12} \quad m_{22} & & & \\ & & & m_{11} & & & \end{array} \right\rangle$$

where

$$m_{ij} \geqslant m_{ij-1} \geqslant m_{i+1j}, \qquad j = 1, 2, \ldots, N, \qquad i = 1, 2, \ldots, N-1 \quad (1)$$

is called a Gel'fand pattern corresponding to $[\lambda]$. The relations (1) are usually referred to as "betweenness relations'. The set (Δ) of numbers $\Delta_1, \Delta_2, \ldots, \Delta_N$ where

$$\Delta_i = \sum_{j=1}^{i} m_{ji} - \sum_{j=1}^{i-1} m_{ji-1}, \qquad i = 1, 2, \ldots, N$$

is called the weight of the Gel'fand pattern $\binom{[m]}{(m)}$. As (m) varies in accordance with the betweenness relations (1), we get several distinct Gel'fand patterns corresponding to $[m]$.

2. Weyl patterns

Corresponding to every Gel'fand pattern we can write a Weyl pattern by the following rule. We take the Young pattern corresponding to the partition $[m]$ and fill up the first row therein by m_{11} ones, $m_{12}-m_{11}$ twos, ..., $m_{1i}-m_{1i-1}$ i's, ..., $m_{1N}-m_{1N-1}$ N's. Then, we fill up the second row by m_{22} twos, $m_{23}-m_{22}$ threes, ..., $m_{2N}-m_{2N-1}$ N's, and so on. Finally in the Nth row we put m_{NN} N's. In the construction of the Weyl patterns corresponding to $[m]$, the numbers $1, 2, \ldots, N$ must occur in a non-decreasing order as we go from left to right in any row and in an increasing order in any column. Given a Weyl pattern, we can write down the corresponding Gel'fand pattern. The correspondence between the Gel'fand patterns and Weyl patterns is thus one-to-one (Baird and Biedenharn 1963). According to the definition of the weight of a Gel'fand pattern one can conclude that the total number of ones, total number of twos, etc, occurring in the corresponding Weyl pattern are respectively, the components of the weight (Δ) of the Gel'fand pattern under consideration (Delaney and Gruber 1969).

If $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0$ and if $\lambda_1+\lambda_2+\ldots+\lambda_N = n$, the Young tableaux obtained by filling the Young pattern $[\lambda]$ with numbers $1, 2, \ldots, n$ without repetition such that each row as well as each column contains the numbers in an increasing order are called standard Young tableaux (Hamermesh 1962). The number of such Young tableaux is denoted by $f_{[\lambda]}$ and is equal to

$$\frac{n!\Pi_{i<k}(l_i-l_k)}{l_1!l_2!\ldots l_n!}$$

where $l_i = \lambda_i+n-i$ $(i = 1, 2, \ldots, n)$.

We denote the number of Gel'fand patterns of $[m]$ corresponding to any given weight (Δ) by the symbol $\mathscr{m}([m],(\Delta))$. This is known as the multiplicity of the weight (Δ) in the irreducible representation $[m]$ of U(N). Many techniques exist for the evaluation of such multiplicities (Delaney and Gruber 1969, Blaha 1969, King and Plunkett 1972).

Theorem. $\mathscr{m}([m],(\Delta)) \leqslant f_{[m]}$.

Proof. Since $m_{1N}+m_{2N}+\ldots+m_{NN} = n$, we have $\Delta_1+\Delta_2+\ldots+\Delta_N = n$. If each Δ_i is 1 $(i = 1, 2, \ldots, N)$ and consequently $N = n$, the Weyl patterns corresponding to the Gel'fand patterns (m) coincide with the standard Young tableaux corresponding to $[m]$. The corresponding Gel'fand patterns are called special Gel'fand patterns. In this case we have $\mathscr{m}([m],(1^n)) = f_{[m]}$ as given by Blaha (1969).

More generally, corresponding to a given weight, we write the n symbols used in the Weyl patterns in the form given below:

$$(S): \qquad (\underbrace{11\ldots11}_{\Delta_1}\ \underbrace{22\ldots2}_{\Delta_2}\ \ldots\ \underbrace{ii\ldots i}_{\Delta_i}\ldots).$$

From any Weyl pattern corresponding to a weight (Δ), we can construct a unique standard Young tableau in the following way.

As we go from left to right in the successive rows of a Weyl pattern, we replace the first '1' we encounter by 1, the second '1' by 2, ..., the Δ_1th '1' by Δ_1; the first '2' we encounter by Δ_1+1, the second '2' by $\Delta_1+2, \ldots$, the Δ_2th '2' by $\Delta_1+\Delta_2, \ldots$ and so on.

According to this convention two different Weyl patterns give rise to two different standard Young tableaux. The standard Young tableaux so obtained for all the Weyl patterns corresponding to a weight (Δ) form a subset Σ of the set Ω of the $f_{[m]}$ standard Young tableaux corresponding to $[m]$. It may be observed that Σ need not always be a proper subset of Ω. This proves that $m([m],(\Delta)) \leqslant f_{[m]}$.

Amongst others, Blaha (1969) has tabulated some results for weight multiplicities.

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